

Thermal Corrections to π - π Scattering Lengths in the Linear Sigma Model

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In this article we address the problem of getting the temperature dependence of the π - π scattering lengths in the frame of the linear sigma model. Using the real time formalism, we calculate all the relevant one loop diagrams. The temperature corrections are only considered in the pion sector, due to the Boltzmann suppression for heavier fields like the sigma meson. From this analysis we obtain the thermal behavior of the s-waves scattering lengths a_0^0 and a_0^2 associated to isospin $I = 0$ and $I = 2$, respectively. If we normalize with the zero temperature value it turns out that $\frac{a_0^0(T)}{a_0^0}$ grows with temperature, whereas the opposite occurs with $\frac{a_0^2(T)}{a_0^2}$. Finally we compare our results with other determinations of the scattering lengths based on the Nambu-Jona-Lasinio model and chiral perturbation theory.

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I. INTRODUCTION

The discussion of the thermal evolution of π - π scattering lengths turns out to be a relevant problem in the context of heavy ion collisions. In fact, we know that in the central rapidity region, precisely where the quark-gluon plasma is expected to be created, a big amount of thermalized pions will be produced. Those pions will interact among themselves and the π - π scattering lengths are crucial parameters in order to describe the scattering amplitudes. At zero temperature the π - π scattering lengths were first measured by Rosellet et al [1]. A recent review about the present status of the experimental measurements of π - π scattering lengths and their comparison with different theoretical approaches can be found in [2].

In this article we will present a detailed calculation of the thermal corrections to the π - π scattering lengths in the frame of the linear sigma model [3]. As it is well known, the linear sigma model is an effective, renormalizable [4], low energy description of hadron dynamics. Our calculations will be done using the real time formalism at the one loop level.

II. LINEAR SIGMA MODEL AND π - π SCATTERING

The linear sigma model in the phase where the chiral symmetry is broken is given by the lagrangian below, where $v = \langle \sigma \rangle$ is the vacuum expectation value of the

scalar field σ . The idea is to define a new field s such that $\sigma = s + v$. Obviously $\langle s \rangle = 0$. ψ corresponds to an isospin doublet associated to the nucleons, $\vec{\pi}$ denotes the pion isotriplet field and $c\sigma$ is the term that breaks explicitly the $SU(2) \times SU(2)$ chiral symmetry. ϵ is a small dimensionless parameter. It is interesting to remark that all fields in the model have masses determined by v . In fact, the following relations are valid: $m_\psi = gv$, $m_\pi^2 = \mu^2 + \lambda^2 v^2$ and $m_\sigma^2 = \mu^2 + 3\lambda^2 v^2$. Perturbation theory at the tree level allows us to identify the pion decay constants as $f_\pi = v$. This model has been considered in the context of finite temperature by several authors, discussing the thermal evolution of masses, $f_\pi(T)$, the effective potential, etc [5], [6], [7], [8].

$$\begin{aligned} \mathcal{L} = & \bar{\psi}[i\gamma^\mu \partial_\mu - m_\psi - g(s + i\vec{\pi} \cdot \vec{\tau}\gamma_5)]\psi + \frac{1}{2}[(\partial\vec{\pi})^2 + m_\pi^2 \vec{\pi}^2] \\ & + \frac{1}{2}[(\partial\sigma)^2 + m_\sigma^2 s^2] - \lambda^2 v s(s^2 + \vec{\pi}^2) - \frac{\lambda^2}{4}(s^2 + \vec{\pi}^2)^2 \\ & + (\epsilon c - v m_\pi^2)s \end{aligned} \quad (1)$$

Since our idea is to use the linear sigma model for calculating π - π scattering lengths, let us remind briefly the formalism. The scattering amplitude has the general form

$$\begin{aligned} T_{\alpha\beta;\delta\gamma} = & A(s, t, u)\delta_{\alpha\beta}\delta_{\delta\gamma} + A(t, s, u)\delta_{\alpha\gamma}\delta_{\beta\delta} \\ & + A(u, t, s)\delta_{\alpha\delta}\delta_{\beta\gamma}. \end{aligned} \quad (2)$$

where the $\alpha, \beta, \gamma, \delta$ denote isospin components.

By using appropriate projection operators, it is possible to find the following isospin dependent scattering amplitudes

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$$T^0 = 3A(s, t, u) + A(t, s, u) + A(u, t, s) \quad (3)$$

$$T^1 = A(t, s, u) - A(u, t, s) \quad (4)$$

$$T^2 = A(t, s, u) + A(u, t, s), \quad (5)$$

where T^I denotes a scattering amplitude in a given isospin channel.

As it is well known [9], the isospin dependent scattering amplitude can be expanded in partial waves T_ℓ^I .

$$T_\ell^I(s) = \frac{1}{64\pi} \int_{-1}^1 d(\cos\theta) P_\ell(\cos\theta) T^I(s, t, u) \quad (6)$$

Below the inelastic threshold the partial scattering amplitudes can be parametrized as [10].

$$T_\ell^I = \left(\frac{s}{s - 4m_\pi^2}\right)^{\frac{1}{2}} \frac{1}{2i} (e^{2i\delta_\ell^I(s)} - 1), \quad (7)$$

where δ_ℓ is a phase-shift in the ℓ channel. The scattering lengths are important parameters in order to describe low energy interactions. In fact, our last expression can be expanded according to

$$\Re(T_\ell^I) = \left(\frac{p^2}{m_\pi^2}\right)^\ell (a_\ell^I + \frac{p^2}{m_\pi^2} b_\ell^I + \dots) \quad (8)$$

The parameters a_ℓ^I and b_ℓ^I are the scattering lengths and scattering slopes, respectively. In general, the scattering lengths obey $|a_0| > |a_1| > |a_2| \dots$. If we are only interested in the scattering lengths a_0^I , it is enough to calculate the scattering amplitude T^I in the static limit, i.e. when $s \rightarrow 4m_\pi^2$, $t \rightarrow 0$ and $u \rightarrow 0$

$$a_0^I = \frac{1}{32\pi} T^I(s \rightarrow 4m_\pi^2, t \rightarrow 0, u \rightarrow 0) \quad (9)$$

III. PION-PION SCATTERING AMPLITUDES

The diagrams shown in figure 1, where the continuum line denotes a pion, and the dashed line a sigma meson, contribute to the π - π scattering amplitude. The diagram with a sigma exchanged meson has to be considered also in the crossed t and u channels.

From these diagrams it is possible to get the results shown in table I. The isospin dependent scattering amplitudes at the tree level have the form

	Experimental Results	Chiral Perturbation Theory	Linear Sigma Model
a_0^0	0.26 ± 0.05	$\frac{7m_\pi^2}{32\pi f_\pi^2} = 0.16$	$\frac{10m_\pi^2}{32\pi f_\pi^2} = 0.22$
b_0^0	0.25 ± 0.03	$\frac{m_\pi^2}{4\pi f_\pi^2} = 0.18$	$\frac{49m_\pi^2}{128\pi f_\pi^2} = 0.27$
a_0^2	-0.028 ± 0.012	$\frac{-m_\pi^2}{16\pi f_\pi^2} = -0.044$	$\frac{-m_\pi^2}{16\pi f_\pi^2} = -0.044$
b_0^2	-0.082 ± 0.008	$\frac{-m_\pi^2}{8\pi f_\pi^2} = -0.089$	$\frac{-m_\pi^2}{8\pi f_\pi^2} = -0.089$
a_1^1	0.038 ± 0.002	$\frac{m_\pi^2}{24\pi f_\pi^2} = 0.030$	$\frac{m_\pi^2}{24\pi f_\pi^2} = 0.030$
b_1^1	—	0	$\frac{m_\pi^2}{48\pi f_\pi^2} = 0.015$

TABLE I: Comparison between the experimental values [1], first order prediction from chiral perturbation theory [11] and our results at the tree level.

$$T^0(s, t, u) = -10\lambda^2 - \frac{12\lambda^4 v^2}{s - m_\sigma^2} - \frac{4\lambda^4 v^2}{t - m_\sigma^2} - \frac{4\lambda^4 v^2}{u - m_\sigma^2} \quad (10)$$

$$T^1(s, t, u) = \frac{4\lambda^4 v^2}{u - m_\sigma^2} - \frac{4\lambda^4 v^2}{t - m_\sigma^2} \quad (11)$$

$$T^2(s, t, u) = -4\lambda^2 - \frac{4\lambda^4 v^2}{t - m_\sigma^2} - \frac{4\lambda^4 v^2}{u - m_\sigma^2}. \quad (12)$$

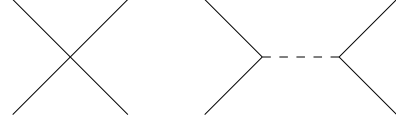


FIG. 1: Tree level diagrams

Note that, the linear sigma model is in a better agreement with the experimental results than first order chiral perturbation theory.

IV. ONE LOOP THERMAL CORRECTIONS FOR SCATTERING LENGTHS

For our calculation of the thermal corrections to the scattering lengths, we will use the real time formalism. At the one loop level it is enough to use the Dolan-Jackiw propagators [12]. A general review about the real time formalism, beyond the one loop level can be found in [13], [14]. In our case the most relevant thermal contributions will be related to the pion sector, due to the Boltzmann suppression in the case of the sigma meson and/or nucleons. Therefore, for the pion propagators we will use

$$\Delta(k_0, \vec{k}, m_\pi) \delta_{\alpha\beta} = \left(\frac{i}{k_0^2 - \vec{k}^2 - m_\pi^2 + i\epsilon} + 2\pi n_B(|k_0|) \delta(k_0^2 - \vec{k}^2 - m_\pi^2) \right) \delta_{\alpha\beta}, \quad (13)$$

where $n_B(x) = \frac{1}{e^{\beta x} - 1}$ is the Bose-Einstein distribution. For the σ meson propagator, we will restrict ourselves to the $T = 0$ case

$$\Delta(k_0, \vec{k}, m_\sigma) = \frac{i}{k_0^2 - \vec{k}^2 - m_\sigma^2 + i\epsilon}. \quad (14)$$

There are many diagrams that contribute to the pion scattering amplitude at the one loop level. For each one of these diagrams we have to add also the corresponding crossed t and u channel diagrams. In figure 2 we have shown only the s channel contribution.

We will give the analytic expression only for the first two diagrams (a,b) shown in figure 2. It should be noticed that, when it corresponds, symmetry factors, isospin index contractions and multiplicity factors should be included.

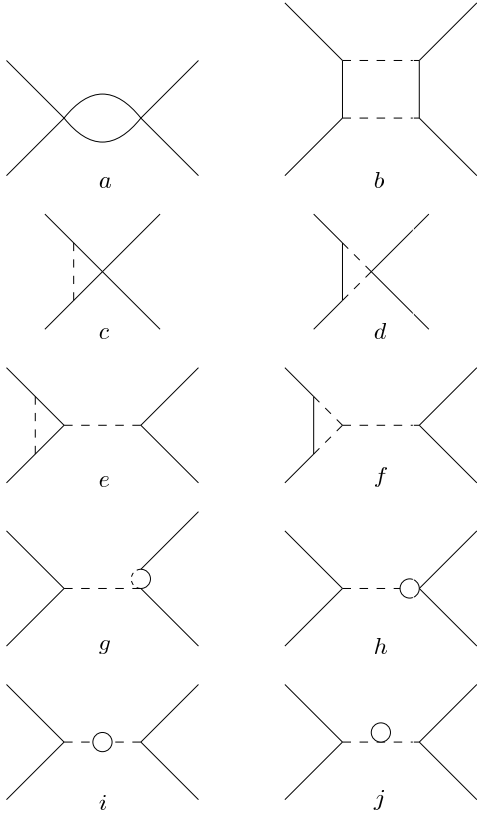


FIG. 2: Relevant one loop diagrams

$$iM_a = -2\lambda^4 (7\delta_{\alpha\beta}\delta_{\gamma\delta} + 2\delta_{\alpha\gamma}\delta_{\beta\delta} + 2\delta_{\alpha\delta}\delta_{\beta\gamma}) \times \int \frac{d^4 k}{(2\pi)^4} \Delta(k_0, \vec{k}, m_\pi) \Delta(k_0 - 2m_\pi, \vec{k}, m_\pi) \quad (15)$$

$$iM_b = 16\lambda^8 v^4 \left(\frac{i}{4m_\pi^2 - m_\sigma^2} \right) \delta_{\alpha\beta} \delta_{\gamma\delta} \int \frac{d^4 k}{(2\pi)^4} [\Delta(k_0, \vec{k}, m_\sigma) \Delta(k_0 + m_\pi, \vec{k}, m_\pi) \Delta(k_0 - m_\pi, \vec{k}, m_\pi)] \quad (16)$$

In some of these diagrams we have to deal with terms $\delta^2(k)$, and $\frac{\delta(k)}{k}$. In those cases the following identity is useful [15]

$$\frac{1}{N!} \left(\frac{i\partial}{\partial m^2} \right)^N \Delta = \Delta^{N+1}. \quad (17)$$

Actually this expression can be proved in the full matrix formalism of ThermoField Dynamics. However, for some diagrams, we checked our results using also the Matsubara (imaginary time) formalism.

When dealing with integrals of the following shape,

$$I = 2\pi i \int dk_0 F(k_0) \left(\frac{\partial}{\partial m_\pi^2} \left[\frac{n_B(|k_0|)}{2w_k} \right] (\delta(k_0 - w_k) + \delta(k_0 + w_k)) + \left(\frac{n_B(|k_0|)}{2w_k} \right) \frac{\partial}{\partial m_\pi^2} [\delta(k_0 - w_k) + \delta(k_0 + w_k)] \right) \quad (18)$$

where $F(k_0)$ is an arbitrary function and $w_k = \sqrt{\vec{k}^2 + m_\pi^2}$, we can integrate by parts getting $I = I_{w_k} + I_{-w_k}$, where

$$I_{w_k} = 2\pi \lim_{k_0 \rightarrow w_k} \left(\frac{i \left(\frac{dF(k_0)}{dk_0} \right) n_B(|k_0|)}{4w_k^2} - i \frac{F(k_0) \text{sgn}(k_0) \left(\text{csch}\left(\frac{|k_0|}{2T}\right) \right)^2}{16w_k^2 T} - i \frac{F(w_k) n_B(w_k)}{4w_k^3} \right) \quad (19)$$

and

$$I_{-w_k} = 2\pi \lim_{k_0 \rightarrow -w_k} \left(\frac{-i \left(\frac{dF(k_0)}{dk_0} \right) n_B(|k_0|)}{4w_k^2} + i \frac{F(k_0) \text{sgn}(k_0) \left(\text{csch}\left(\frac{|k_0|}{2T}\right) \right)^2}{16w_k^2 T} - i \frac{F(-w_k) n_B(w_k)}{4w_k^3} \right). \quad (20)$$

After taking all the diagrams into account, we find the following expression for the one loop thermal corrections to the π - π scattering amplitudes

$$\begin{aligned}
A(s, t, u)_T = & -\lambda^4(14 + 80\lambda^2 v^2 (\frac{1}{4m_\pi^2 - m_\sigma^2})) \\
& + 48\lambda^4 v^4 (\frac{1}{4m_\pi^2 - m_\sigma^2})^2 A(w_k, T) - 8\lambda^4 B(w_k, T) \\
& - 12\lambda^6 v^2 (\frac{1}{4m_\pi^2 - m_\sigma^2})^2 C(w_k, T) \\
& + \lambda^6(16v^2 + 32\lambda^2 v^4 (\frac{1}{4m_\pi^2 - m_\sigma^2})) \\
& \times [D^1(m_\pi, m_\sigma, w_k, T) + D^1(-m_\pi, m_\sigma, w_k, T)] \\
& + 32\lambda^6 v^2 [E_0^1(m_\pi, m_\sigma, w_k, T) + E_0^1(-m_\pi, m_\sigma, w_k, T)] \\
& + 32\lambda^8 v^4 [E_1^1(m_\pi, m_\sigma, w_k, T) + E_1^1(-m_\pi, m_\sigma, w_k, T)] \\
& + 8\lambda^6 v^2 (\frac{1}{4m_\pi^2 - m_\sigma^2}) \times [F_0^1(m_\pi, m_\sigma, w_k, T) \\
& + F_0^1(-m_\pi, m_\sigma, w_k, T)] + \lambda^6 [96\lambda^2 v^4 (\frac{1}{4m_\pi^2 - m_\sigma^2}) \\
& + 16v^2] F_1^1(m_\pi, m_\sigma, w_k, T), \tag{21}
\end{aligned}$$

$$\begin{aligned}
A(t, s, u)_T = & -4\lambda^4 A(w_k, T) - \lambda^4(11 + 80\lambda^2 v^2 (\frac{1}{-m_\sigma^2})) \\
& + 48\lambda^4 v^4 (\frac{1}{-m_\sigma^2})^2 B(w_k, T) + 4\lambda^6 v^2 (\frac{1}{-m_\sigma^2}) \\
& \times (\frac{4}{m_\sigma^2 - m_\pi^2} + \frac{3}{m_\sigma^2}) C(w_k, T) \\
& + 16\lambda^6 v^2 [D^1(m_\pi, m_\sigma, w_k, T) + D^1(-m_\pi, m_\sigma, w_k, T)] \\
& + 16\lambda^8 v^4 [D^2(m_\pi, m_\sigma, w_k, T) + D^2(-m_\pi, m_\sigma, w_k, T)] \\
& + \lambda^6(32v^2 + 32\lambda^2 v^4 (\frac{1}{-m_\sigma^2})) \\
& \times [E_0^1(m_\pi, m_\sigma, w_k, T) + E_0^1(-m_\pi, m_\sigma, w_k, T)] \\
& + 16\lambda^8 v^4 [E_0^2(m_\pi, m_\sigma, w_k, T) + E_0^2(-m_\pi, m_\sigma, w_k, T)] \\
& + \lambda^6(48\lambda^2 v^4 (\frac{1}{-m_\sigma^2}) + 8v^2) [F_0^2(m_\pi, m_\sigma, w_k, T) \\
& + F_0^2(-m_\pi, m_\sigma, w_k, T)] \\
= & A(u, t, s)_T, \tag{22}
\end{aligned}$$

where we have introduced the following definitions

$$A(w_k, T) = \frac{-1}{(2\pi)^2} \int d|\vec{k}| \frac{n_B(w_k)}{w_k}, \tag{23}$$

$$B(w_k, T) = - \int \frac{d^3 k}{(2\pi)^3} (\frac{n_B(w_k)}{2w_k^3} + \frac{(csch(\frac{w_k}{2T}))^2}{8T w_k^2}), \tag{24}$$

$$C(w_k, T) = \int \frac{d^3 k}{(2\pi)^3} \frac{n_B(w_k)}{w_k}, \tag{25}$$

$$\begin{aligned}
D^n(x, y, w_k, T) = & - \int \frac{d^3 k}{(2\pi)^3} \frac{n_B(w_k)}{w_k} \\
& \times [\frac{1}{f^n(x, y)} (\frac{1}{4x^2 + 4xw_k})] \tag{26}
\end{aligned}$$

$$\begin{aligned}
E_m^n(x, y, w_k, T) = & \int \frac{d^3 k}{(2\pi)^3} [\frac{(x + w_k)n_B(w_k)}{2w_k^2 f^{n+1}(x, y) f^m(-x, y)} \\
& (\frac{(csch(\frac{w_k}{2T}))^2}{16T w_k^2} + \frac{n_B(w_k)}{4w_k^3}) (\frac{1}{f^n(x, y) f^m(-x, y)})] \tag{27}
\end{aligned}$$

$$F_m^n(x, y, w_k, T) = - \int \frac{d^3 k}{(2\pi)^3} \frac{n_B(w_k)}{w_k} [\frac{1}{f^n(x, y) f^m(-x, y)}], \tag{28}$$

and where

$$f^n(x, y) = (2x^2 + 2xw_k - y^2)^n. \tag{29}$$

V. NUMERICAL RESULTS

The thermal corrections must be calculated numerically. The different parameters in our expressions are renormalized at $T = 0$, since thermal corrections do not add new ultraviolet divergencies. The linear sigma model, excluding the nucleons, has three parameters: m_π^2 , f_π and λ^2 . The first two parameters, m_π^2 and f_π , are given by experiments and the third one is a free parameter. Notice that f_π is related to the vacuum expectation value v . In fact, at the tree level $f_\pi = v$. The three parameters are not independent. If instead of f_π we use the vacuum expectation value v and consider a mass of the sigma meson $m_\sigma = 700$ MeV, we have $\lambda^2 = 7$, $v = 90$ MeV; if $\lambda^2 = 5.6$, $v = 120$ MeV [16]. We know, however, that the mass of the sigma meson is about $m_\sigma = 550$ MeV [17]. Therefore, we need to find new values for λ and v associated to the new lower mass of the sigma meson. We found $\lambda^2 = 4.26$ and $v = 89$ MeV, following the philosophy presented in [16]. The authors used the Padé approximant method, but they also suggested an analytic approach in order to find the variation of one parameter if the other two change as a consequence of one loop corrections. The values given above for λ^2 and v were found following this procedure.

The scattering lengths, including our thermal corrections are given by

$$a_0^0(T) = 0.217 + \frac{3A(s, t, u)_T + 2A(t, s, u)_T}{32\pi} \tag{30}$$

$$a_0^2(T) = -0.041 + \frac{2A(t, s, u)_T}{32\pi}. \tag{31}$$

The behavior of the normalized scattering lengths $\frac{a_0^0(T)}{a_0^0}$ and $\frac{a_0^2(T)}{a_0^2}$ are shown in figure 3. Notice that $a_0^1(T)$ vanishes identically. We normalized with the zero temperature values of the scattering lengths, according to a two loop perturbation calculation [18]. It is interesting to remark that $a_0^0(T)$ grows with temperature and the same occurs with the $a_0^2(T)$. Similar calculations have

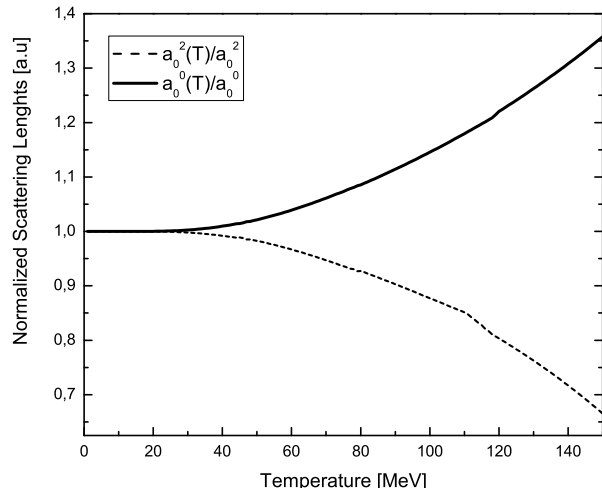


FIG. 3: Scattering lengths normalized to $T = 0$

been done in the context of the Nambu-Jona-Lasinio model [19] and in the frame of chiral perturbation theory [20]. The results from the Nambu-Jona-Lasinio analysis do not agree with our conclusions. In this approach both scattering lengths remain almost constant, but with the same growing tendency. The results from chiral perturbation theory agree quite well with our conclusions for $a_0^2(T)$. Nevertheless, $a_0^0(T)$ is almost constant in this approach. In [21], a different determination of $a_0^0(T)$ based on the Heat Kernel expansion technique applied to the linear sigma model, is presented. We agree with their results.

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